SYMMETRIZED *(p,h)*-CONVEXITY AND SOME HERMITE-HADAMARD-TYPE INEQUALITIES

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SUMMARY

This paper introduces symmetrized *(p,h)*-convex functions and establishes some Hermite-Hadamard-type inequalities for the new class of functions.

Keywords: *symmetrized (p,h)-convexity, (p,h)-convexity, p-convexity, h-convexity, harmonically convexity, Hermite-Hadamard inequality.*

1. INTRODUCTION

A function f is said to be *convex* on a real interval I if

$$f(\lambda x+(1-\lambda)y)\le \lambda f(x)+(1-\lambda)f(y),$$

for all $x,y\in I$ and $\lambda\in[0,1]$. An estimate of the (integral) mean value of a continuous convex function is known as Hermite-Hadamard inequality (Hadamard, 1893, Hermite, 1883). Precisely, if f is a convex function on a real interval I and $a,b\in I$ with $a<b$ then

$$f\left(\frac{a+b}{2}\right)\le \frac{1}{b-a}\int_a^b f(x)dx\le \frac{f(a)+f(b)}{2}.$$

This inequality is one of the most useful inequalities in mathematical analysis. For new proofs, noteworthy extensions, generalizations, and numerous applications of this inequality, see e.g., (Dragomir & Pearce, 2003, Mitrinović & Lacković, 1985).

In recent years, convexity has been generalized and extended in various aspects using new and different concepts. These researches led to the appearance of several Hermite-Hadamard-type inequalities. Among these generalizations, two interesting concepts are *p*-convex functions (Zhang & Wan, 2007) and *h*-convex functions (Varošanec, 2007). They are defined as follows:

Definition 1.1 (Zhang & Wan, 2007). Let $I\subset(0,\infty)$ be an interval and $p\in\mathbb{R}\setminus\{0\}$. We say that $f:I\to\mathbb{R}$ is a *p*-convex function if

$$f\left([\alpha x^p+(1-\alpha)y^p]^{1/p}\right)\le \alpha f(x)+(1-\alpha)f(y),$$

for all $x,y\in I$ and $\alpha\in[0,1]$.

Definition 1.2 (Varošanec, 2007). Let I,J be two real intervals with $J\supseteq(0,1)$ and let $h:J\to\mathbb{R}$ be a non-negative and non-zero function. We say that $f:I\to\mathbb{R}$ is a *h*-convex function if f is non-negative and

$$f(\alpha x+(1-\alpha)y)\le h(\alpha)f(x)+h(1-\alpha)f(y),$$

for all $x,y\in I$ and $\alpha\in(0,1)$.

These functions are continuously generalized to (*p,h*)-convex functions by Fang & Shi (2014):

Definition 1.3 (Fang & Shi, 2014). Let $I\subset(0,\infty)$ and $J\supseteq(0,1)$ be two real intervals, $p\in\mathbb{R}\setminus\{0\}$, and let $h:J\to\mathbb{R}$ be a non-negative and non-zero function. We say that $f:I\to\mathbb{R}$ be a *(p,h)*-convex function if f is non-negative and

$$f\left([\alpha x^p+(1-\alpha)y^p]^{1/p}\right)$$
$$\le h(\alpha)f(x)+h(1-\alpha)f(y),$$

for all $x,y\in I$ and $\alpha\in(0,1)$.

In (Fang & Shi, 2014), the authors also derived Hermite-Hadamard-type inequalities for the class of *(p,h)*-convex functions as follows:

$$\frac{1}{2h\left(\frac{1}{2}\right)}f\left(\left[\frac{a+b}{2}\right]^{1/p}\right)\le \frac{p}{b^p-a^p}\int_a^b x^{p-1}f(x)dx$$
$$\le (f(a)+f(b))\int_0^1 h(t)dt. \quad (1.1)$$

In particular, if $p=-1$, *(p,h)*-convex functions become harmonically *h*-convex functions which are studied firstly by Noor & et al. (2015).

A significant generalization of *h*-convex functions is symmetrized *h*-convex functions, which are introduced by Dragomir (2016). To give this concept, the authors considered a symmetrical transform $\overline{f}$ of f on $[a,b]$ which is defined by

$$\overline{f}(x):=\frac{1}{2}[f(x)+f(a+b-x)].$$

Then we say $f:[a,b]\to[0,\infty)$ is symmetrized *h*-convex function on $[a,b]$ if $\overline{f}$ is *h*-convex on $[a,b]$. Moreover, if h integrable on $[0,1]$ and f integrable on $[a,b]$, Hermite-Hadamard-type inequality for symmetrized *h*-convex function is given as follows (Dragomir, 2016):

$$\frac{1}{2h\left(\frac{1}{2}\right)}f\left(\frac{a+b}{2}\right)\le \frac{1}{b-a}\int_a^b f(x)dx$$
$$\le \left[f(a)+f(b)\right]\int_0^1 h(x)dx. \quad (1.2)$$

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