GENERALIZATIONS OF YOUNG-TYPE INEQUALITIES VIA QUADRATIC INTERPOLATION

Phan Thi Minh[1], Tran Quynh Mai[2], Duong Quoc Huy[3]

Received Date: 15/5/2022; Revised Date: 10/10/2022; Accepted for Publication: 11/10/2022

SUMMARY

In this paper, we give some new improvements of the famous works of F. Kittaneh, Y. Manasrah about Young's inequalities published on the J. Math. Anal. Appl. (2010) and Linear Multilinear Algebra (2011) via the theory of quadratic interpolations. As applications, we also establish corresponding inequalities for matrix and operator versions.

Keywords: *Young inequality, Convexity, Positive operator, Positive definite Matrix.*

1. INTRODUCTION

The classical Young inequality for scalars states that for all $a,b>0$ and $\nu\in[0,1]$, we have

$$(1-\nu)a+\nu b\ge a^{1-\nu}b^{\nu}, \qquad (1)$$

with equality if and only if $a=b$. The inequality (1) is also known in the literature as the weighted arithmetic-geometric mean inequality.

One of the most striking refinements and reverses of (1) was established by Kittaneh and Manasrahin (Kittaneh & Manasrah, 2010, 2011) as

$$\begin{aligned} r_0(\sqrt{a}-\sqrt{b})^2+a^{1-\nu}b^{\nu} \\ \le (1-\nu)a+\nu b \qquad (2)\\ \le a^{1-\nu}b^{\nu}+R_0(\sqrt{a}-\sqrt{b})^2, \end{aligned}$$

here and hereafter $r_0=\min\{\nu,1-\nu\}$ and $R_0=\max\{\nu,1-\nu\}$. The equality sign in (2) also happen when $a=b$.

Relying on the proof of these inequalities, we can not see fully the source of non-negative quantities $r_0(\sqrt{a}-\sqrt{b})^2$ and $R_0(\sqrt{a}-\sqrt{b})^2$ in (2). However, this will be explained more clearly when applying the well-known Jensen-type inequalities:

$$\begin{aligned} r_0\left(\frac{\varphi(0)+\varphi(1)}{2}-\varphi\left(\frac{1}{2}\right)\right) \\ \le (1-\nu)\varphi(0)+\nu\varphi(1)-\varphi(\nu) \qquad (3)\\ \le R_0\left(\frac{\varphi(0)+\varphi(1)}{2}-\varphi\left(\frac{1}{2}\right)\right) \end{aligned}$$

for the convex function $\varphi(\nu)=a^{1-\nu}b^{\nu}$ with $a,b>0$ and $\nu\in[0,1]$. The inequalities in (3) were proposed by Dragomir (Dragomir, 2006). The approach via the convexity of functions not only explain the source of (2), but also provide a unified method for many recent results related to Young's inequality, its refinements and reverses, see (ChoiKrnić, & Pecarić, 2017) and references therein.

In this article, we proceed to use the convexity of functions to give a more further refinements of (2). To see this idea clearly, we focus primarily on the case $a\ne b$. The left-hand inequality in (2) is then equivalent to

$$f(\nu):=\frac{(1-\nu)a+\nu b-a^{1-\nu}b^{\nu}}{(\sqrt{a}-\sqrt{b})^2}\ge r_0 \text{ for all } \nu\in[0,1]$$

Evidently, f is a concave function on $[0,1]$ because, for all $\nu\in[0,1]$,

$$f''(\nu)=-a^{1-\nu}b^{\nu}\left(\frac{\ln a-\ln b}{\sqrt{a}-\sqrt{b}}\right)^2\le 0.$$

It is easy to see that $r_0=\min\{\nu,1-\nu\}$ is a concave function on $[0,1]$; moreover, it is a concave interpolation of f on $[0,1]$ with nodes $0,\frac{1}{2},1$. However, if we only consider the interval $[0,\frac{1}{2}]$, the function $r_0=\nu$ is only a linear interpolation of the function f. This shows that the function given by

$$g(\nu):=\frac{(1-\nu)a+\nu b-a^{1-\nu}b^{\nu}}{(\sqrt{a}-\sqrt{b})^2}-\nu$$

is still concave on $[0,\frac{1}{2}]$. This, together with the

above mentioned idea, motivates us to find a concave interpolation h of the function g on the interval $[0,\frac{1}{2}]$ such that, for all $\nu\in[0,\frac{1}{2}]$,

$$h(\nu)\ge 0 \quad \text{and} \quad g(\nu)\ge h(\nu).$$

In the present paper, we give a concave interpolation of the quadratic form

[1] *Phan Chu Trinh hight school, Cu Jut, Dak Nong;*
[2] *Office of Academic Affairs, Tay Nguyen University, Dak Lak;*
[3] *Faculty of Natural Science and Technology, Tay Nguyen University, Dak Lak.*
Corresponding author: Duong Quoc Huy; Tel: 0905616183; Email: duongquochuy@ttn.edu.vn